# ÉTALE COHOMOLOGY OF A DM CURVE-STACK WITH COEFFICIENTS IN $\mathbb{G}_m$

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Abstract. We compute étale cohomology groups $H^i_{\text{ét}}(X, \mathbb{G}_m)$ in several cases, where $X$ is a smooth tame Deligne-Mumford stack of dimension 1 over an algebraically closed field. We have complete results for orbicurves (and, more generally, for twisted nodal curves) and in the case all stabilizers are cyclic; we give partial results and examples in the general case. In particular, we show that if the stabilizers are abelian then $H^2_{\text{ét}}(X, \mathbb{G}_m)$ does not depend on $X$ but only on the underlying orbicurve and on the generic stabilizer.

## 1. Introduction

A classical theorem of Tsen states that a function field $K$ of dimension 1 over an algebraically closed field is quasi-algebraically closed (Theorem 2.4). As a consequence, the étale cohomology groups $H^r_{\text{ét}}(\operatorname{Spec} K, \mathbb{G}_m)$ vanish for $r \geq 1$ (Corollary 2.3). Combining this result with étale cohomology techniques, Milne showed that, for a connected smooth curve $C$ over an algebraically closed field, $H^r_{\text{ét}}(C, \mathbb{G}_m)$ vanishes for $r \geq 2$ ([8], III.2.22(d)). Étale cohomology in low degree with coefficients in $\mathbb{G}_m$ can be interpreted geometrically. It is well-known for a scheme $X$ that $H^1_{\text{ét}}(X, \mathbb{G}_m) \cong \operatorname{Pic}(X)$ ([8], III.4.9). Moreover, for a smooth variety $X$ over a field, $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is isomorphic to the Brauer group $\operatorname{Br}(X)$ ([8], IV.2.15). It follows that computing the groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ contributes to a better understanding of geometric properties of $X$. As an example, the vanishing of $H^2_{\text{ét}}(X, \mathbb{G}_m)$ implies that every gerbe over $X$ banded by a finite group (of order not divided by the characteristic of the base field) is obtained as a finite number of root constructions ([4]). More in general, once we know $H^r_{\text{ét}}(X, \mathbb{G}_m)$, we can use Kummer sequence to compute $H^r_{\text{ét}}(X, \mu_n)$, for all $n$ not divided by the characteristic of the base field.

In this paper we study the groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ where $X$ is a connected smooth tame Deligne-Mumford stack of dimension 1 over an algebraically closed field $k$. Such an $X$ admits a natural map $X \to C$ to its coarse moduli space, which is a connected smooth curve over $k$; moreover the map $X \to C$ factors via an étale gerbe $X \to Y$, where $Y$ is an orbicurve ([1], Appendix A).

We generalize to algebraic stacks the étale cohomology techniques used by Milne and we obtain a description of the groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ (Theorem 4.10). In particular, we show that if the stabilizers are abelian then $H^2_{\text{ét}}(X, \mathbb{G}_m)$ depends only on the orbicurve $Y$ and the generic stabilizer, not on the structure of the gerbe $X \to Y$ (Proposition 4.12). This result suggests to investigate two special cases: when $X = Y$ is an orbicurve (Corollary 4.15) and when the gerbe $X \to Y$ is trivial (Proposition 6.1). The cohomology of orbicurves, and more in general of twisted nodal curves, has been described in [5], Theorem 3.2.3. We include it for completeness, since we give a different proof (Proposition 5.2). We show that, for a twisted nodal curve $Y$, the vanishing of $H^2_{\text{ét}}(Y, \mathbb{G}_m)$ implies that every $G$-gerbe over $Y$ banded by a finite group $G$ (of order not divided by $\operatorname{char} k$), can be obtained as a finite number of root constructions (Section 5.1). This fact has been used in [3] and [5] to study the moduli stack of twisted stable maps and the associated Gromov-Witten invariants.

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Finally, we show with two examples that, in general, the higher cohomology groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ cannot be computed knowing only the base of the gerbe $X \to Y$ and the banding group (Section 7).

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**Notations.** We write $\mathrm{Br}(K)$ for the Brauer group of a field $K$. With the word stack we always mean a Deligne-Mumford algebraic stack in the sense of [6]. All stacks are assumed to be separated and of finite type over the base field. For the notion of tame stack see [1]. An orbicurve is a stack of dimension 1 with trivial generic stabilizer. We denote by $\mathscr{O}^{\text{sh}}_{S,\overline{x}}$ the strictly Henselian local ring of a scheme (or an algebraic stack) $S$ at a geometric point $\overline{x} \to S$. We denote by $\mu_n$ the sheaf defined by the group scheme $\operatorname{Spec} {}^{\mathbb{Z}[t]}/_{(t^n-1)}$.

## 2. Preliminaries

We recall a few classical theorems about quasi-algebraically closed fields and cohomology of finite abelian groups (for more details see [10], chapter IV, and [11], chapter VI).

2.1. **Definition.** A field $K$ is quasi-algebraically closed if every non constant homogeneous polynomial $f(x_1, \dots, x_N)$ of degree $d < N$ with coefficients in $K$ has a non-trivial zero in $K$.

2.2. **Proposition** ([10], IV.3 Corollary 1)**.** *Let $K$ be a quasi-algebraically closed field and let $G$ denote the absolute Galois group $\mathrm{Gal}({}^{K_s}/_K)$, where $K_s$ is a separable closure of $K$. Then*

*(1)* $\mathrm{Br}(K) = 0$*;*
*(2) $H^r(G, T) = 0$ for $r \geq 2$ and for all discrete torsion $G$-modules $T$;*
*(3) $H^r(G, M) = 0$ for $r \geq 3$ and for all discrete $G$-modules $M$.*

2.3. **Corollary.** *Let $K$ be a quasi-algebraically closed field. Then $H^r_{\acute{e}t}(\operatorname{Spec} K, \mathbb{G}_m) = 0$ for $r \geq 1$.*

*Proof.* By [8], III.1.7 and Proposition 2.2,

$$H^r_{\text{ét}}(\operatorname{Spec} K, \mathbb{G}_m) = \begin{cases} \mathrm{Pic}(\operatorname{Spec} K) = 0 & \text{if } r = 1 \\ H^2(G, K_s^*) = \mathrm{Br}(K) = 0 & \text{if } r = 2 \\ H^r(G, K_s^*) = 0 & \text{if } r \geq 3. \end{cases} \qquad \square$$

2.4. **Theorem** ([10], IV.3 Theorem 24)**.** *Let $K$ be a function field of dimension* 1 *over an algebraically closed field $k$. Then $K$ is quasi-algebraically closed.*

2.5. **Theorem** ([10], IV.3 Theorem 27)**.** *Let $K$ be the field of fractions of an Henselian discrete valuation ring $R$ with algebraically closed residue field. Let $\hat{K}$ be the completion of $K$, and assume that $K \subset \hat{K}$ is a separable field extension. Then $K$ is quasi-algebraically closed.*

2.6. Remark. The separability condition in Theorem 2.5 holds in the case $R = \mathscr{O}^{\text{sh}}_{X,\overline{x}}$, with $X$ a scheme of finite type over a field ([8], III.2.22(b)).

2.7. **Theorem** ([11], Theorem 6.2.2)**.** *Let $G = {}^{\mathbb{Z}}/_{d\mathbb{Z}}$ and let $A$ be a discrete $G$-module. Then*

$$H^r(G, A) = \begin{cases} A^G & r = 0 \\ {}^{\ker \mathrm{N}_G}/_{I_G A} & r \equiv 1 \quad (2) \\ {}^{A^G}/_{\operatorname{im} \mathrm{N}_G} & r \equiv 0 \quad (2),\ r > 0, \end{cases}$$

*where $\mathrm{N}_G \colon A \to A$ is the norm $\mathrm{N}_G(a) = \sum_{g \in G} ga$, $A^G$ is the set of elements in $A$ fixed by the $G$-action, and $I_G A$ is the subgroup of $A$ generated by elements $(ga - a)$, with $a \in A$ and $g \in G$.*

2.8. **Corollary.** *Let $G$ be a finite cyclic group acting trivially on $\mathbb{Z}$, then*

$$H^r(G,\mathbb{Z}) = \begin{cases} \mathbb{Z} & r=0 \\ 0 & r\equiv 1 \quad (2) \\ G & r\equiv 0 \quad (2),\ r>0. \end{cases}$$

2.9. **Lemma.** *Let $G$ be a finite group and let $(J^\bullet, d)$ be a cochain complex of abelian groups on which $G$ acts trivially. Then, for $p\ge 0$, $q\ge 1$ and $r=2,\dots,q+1$, the following map is zero*

$$\Phi_r^{p,q}\colon H^p(G,H^q(J^\bullet)) \to H^{p+r}(G,H^{q-r+1}(J^\bullet)).$$

*Proof.* Let assume $r=2$. The map $\Phi_2^{p,q}$ is defined as follows. Let $\xi\in H^p(G,H^q(J^\bullet))$, then $\xi$ is the class of a cycle $f\colon G^p\to H^q(J^\bullet)$. Let $\tilde f\colon G^p\to Z^q(J^\bullet)$ be a lifting of $f$ (we write $Z^q(J^\bullet)$ for cycles in $J^q$). If we denote by $d_G$ the boundary map for the group cohomology, then we have $d_G(f)=0$, hence $d_G(\tilde f)\colon G^{p+1}\to d(J^{q-1})$. In particular, there exists a map $\nu_f\colon G^{p+1}\to J^{q-1}$ such that $d\circ\nu_f = d_G(\tilde f)$. Then $\Phi_2^{p,q}(\xi)=\pi\circ d_G(\nu_f)$, where $\pi\colon Z^q(J^\bullet)\to H^q(J^\bullet)$ is the projection. It is enough to show that $d_G(\tilde f)=0$, because then $\operatorname{im}\nu_f\subset Z^{q-1}(J^\bullet)$ and therefore $\pi\circ d_G(\nu_f)=d_G(\pi\circ\nu_f)=0$. Since the action of $G$ is trivial, for all $(g_1,\dots,g_{p+1})\in G^{p+1}$,

$$d_G(\tilde f)(g_1,\dots,g_{p+1}) = \tilde f(g_2,\dots,g_{p+1}) + \sum_{i=1}^{p}(-1)^i \tilde f(g_1,\dots,g_ig_{i+1},\dots,g_{p+1}) + (-1)^{p+1}\tilde f(g_1,\dots,g_p).$$

If we require $d_G(\tilde f)=0$, then we get a linear system of $n^{p+1}$ equations of the form

$$x_{(g_2,\dots,g_{p+1})} + \sum_{i=1}^{p}(-1)^i x_{(g_1,\dots,g_ig_{i+1},\dots,g_{p+1})} + (-1)^{p+1}x_{(g_1,\dots,g_p)} = 0, \tag{1}$$

in the indeterminates $\left\{ x_{\underline{g}} \,\middle|\, \underline{g}=(g_1,\dots,g_p)\in G^p \right\}$, where $n$ is the order of $G$. Since $d_G(f)=0$, there exists a solution of these equations in $H^q(J^\bullet)$. In particular there is a set $I$ of indeterminates whose values can be chosen freely, and the values of the others are determined by (1). If $I=\emptyset$ then there is only the trivial solution. Notice that $\pi$ preserves relations (1). Hence we can assign values $z_{\underline{g}}\in Z^p(J^\bullet)$ to the indeterminates in $I$ so that $\pi(z_{\underline{g}})=f(\underline{g})$. Then, using equations (1), we find a solution $z_{\underline{g}}\in Z^p(J^\bullet)$ of the system (1). It follows that the map $\tilde f$ defined by $\tilde f(\underline{g})=z_{\underline{g}}$ is a lifting of $f$ such that $d_G(\tilde f)=0$. Hence $\Phi_2^{p,q}=0$, for all $p\ge 0$ and $q\ge 1$.

The maps $\Phi_r^{p,q}$ are defined recursively. Let $\xi\in H^p(G,H^q(J^\bullet))$, and consider $\Phi_r^{p,q}(\xi)$. Let $\varphi\colon G^{p+r}\to H^{q-r+1}(J^\bullet)$ be a cycle that represents $\Phi_r^{p,q}(\xi)$. We know that

$$\Phi_2^{p+r,q-r+1}(\Phi_r^{p,q}(\xi)) = d_G(\pi\circ\nu_\varphi)=0,$$

with $\nu_\varphi$ defined as above, then we set $\Phi_{r+1}^{p,q}(\xi)=\pi\circ\nu_\varphi$. Now we prove the statement by induction on $r$. Assume that $\Phi_r^{p,q}=0$. Let $\xi\in H^p(G,H^q(J^\bullet))$, then $\varphi=\Phi_r^{p,q}(\xi)=0$. It follows that there exists $\rho\colon G^{p+r}\to Z^{q-r}(J^\bullet)$ which is a lifting of $\varphi$. Hence we can take $\nu_\varphi=d_G(\rho)$ and we have

$$\pi\circ\nu_\varphi = \pi\circ d_G(\rho) = d_G(\pi\circ\rho)=0.$$ □

## 3. Setting

Let $X$ be a connected smooth tame Deligne-Mumford stack of dimension 1 over an algebraically closed field $k$. Let $\eta$ be the generic point of $X$ and let $g\colon\eta\to X$ be the inclusion. We denote by $G_0$ the generic stabilizer. Moreover if $\sigma$ is a closed point of $X$ with stabilizer $G_\sigma$, we write $\sigma\colon\sigma\hookrightarrow X$ for the inclusion and we denote by $X(k)$ the set of closed points of $X$. Let $C$ be the coarse moduli space of $X$, we denote by $\pi\colon X\to C$ the natural map.

We begin with some general remarks on the stack $X$ and its coarse moduli space.

3.1. **Proposition.** *The coarse moduli space $C$ is a connected smooth curve over $k$.*

*Proof.* Since $X$ is connected and $\pi$ is surjective, also $C$ is connected. Moreover, there exists an étale cover of $C$ consisting of schemes of the form ${}^{U}\!/\!{}_{G}$, where $U$ is a smooth affine scheme of dimension 1 over $k$ and $G$ is a finite group of order not divided by $\operatorname{char} k$ ([1], Theorem 3.2). By [8], Proposition I.3.24, locally in the étale topology, there exists a linearization of the action of $G$ on $U$. In particular $G \subset k^*$ is a finite subgroup, hence $G = \mu_n$. It follows that, locally in the étale topology, ${}^{U}\!/\!{}_{G} = \operatorname{Spec}(k[t])^{\mu_n} = \operatorname{Spec} k[t^n]$, which is smooth of dimension 1. □

3.2. **Theorem** ([8], III.2.22(d))**.** *Let $C$ a smooth curve over an algebraically closed field. Then*

$$H^r_{\acute{e}t}(C, \mathbb{G}_m) = \begin{cases} \Gamma(C, \mathbb{G}_m) & r = 0 \\ \operatorname{Pic}(C) & r = 1 \\ 0 & r \geq 2. \end{cases}$$

3.3. **Theorem** ([1], A.1)**.** *Let $M$ be a regular Deligne-Mumford stack over a field $k$. Let $\mathscr{G}$ be the closure of the fibers of the inertia stack $I(M)$ over the generic points of irreducible components of the moduli space of $M$. Then there exist a regular Deligne-Mumford stack $Y$ with trivial generic stabilizer and an étale morphism $M \xrightarrow{\pi} Y$, such that $M$ is a $\mathscr{G}$-gerbe banded by $\mathscr{G}$ over $Y$.*

3.4. By Theorem 3.3, $X$ is an étale gerbe over an orbicurve $Y$. It is known that $Y$ is obtained from its coarse moduli space $C$ by applying a finite number of root constructions. Explicitly, there exist distinct points $p_1, \dots, p_N \in C(k)$ and integers $d_1, \dots, d_N \geq 2$, with $N \geq 0$, such that $X = X_1 \times_C \cdots \times_C X_N$, where $X_l = \sqrt[d_l]{{}^{p_l}\!/\!{}_{C}}$, for $l = 1, \dots, N$ (for details on the root construction see [4]). Let $\Sigma = \{\sigma_1, \dots, \sigma_N\}$ be the set of closed points of $X$ corresponding to $p_1, \dots, p_N \in C(k)$. Then $G_\sigma = G_0$ for $\sigma \in X(k)$, $\sigma \notin \Sigma$, and ${}^{G_{\sigma_l}}\!/\!{}_{G_0} = {}^{\mathbb{Z}}\!/\!{}_{d_l\mathbb{Z}}$ for $l = 1, \dots, N$.

3.5. **Proposition.** *Let $\operatorname{Spec} K$ be the generic point of $C$, then $\eta = \operatorname{Spec} K \times \mathrm{B}G_0$.*

*Proof.* By Theorem 3.3, the stack $X$ is a gerbe over $Y$ banded by $\mathscr{G}$. Let $\mathscr{G}_\eta$ be the sheaf over $\operatorname{Spec} K$ induced by $\mathscr{G}$. Notice that $\eta = \operatorname{Spec} K \times_Y X$, hence $\eta \to \operatorname{Spec} K$ is a $\mathscr{G}_\eta$-gerbe banded by $\mathscr{G}_\eta$. Recall that such gerbes are classified by $H^2_{\text{ét}}(\operatorname{Spec} K, \mathscr{Z})$, where $\mathscr{Z}$ is the center of $\mathscr{G}_\eta$ ([7], IV.5.2). Let $\operatorname{Gal}({}^{K_s}\!/\!{}_{K})$ be the absolute Galois group of $K$, then there is an isomorphism

$$H^r_{\text{ét}}(\operatorname{Spec} K, \mathscr{Z}) \cong H^r(\operatorname{Gal}({}^{K_s}\!/\!{}_{K}), M),$$

with $M = \varinjlim_L \mathscr{Z}(\operatorname{Spec} L)$, where the limit is taken over all subfields $L \subset K_s$ that are finite over $K$ ([8], III.1.7). Notice that $M$ is a torsion module, hence, by Proposition 2.2, $H^r(\operatorname{Gal}({}^{K_s}\!/\!{}_{K}), M) = 0$ for $r \geq 2$. In particular $H^2_{\text{ét}}(\operatorname{Spec} K, \mathscr{Z}) = 0$. □

3.6. Remark. The action of $G_0$ over $K^*$ is trivial, since the sheaf $\mathbb{G}_{m,\eta}$ comes from the sheaf $\mathbb{G}_{m,\operatorname{Spec} K}$ (which is equivariant under the action of $G_0$). It follows that $\pi_{\eta *}\mathbb{G}_{m,\eta} = \mathbb{G}_{m,\operatorname{Spec} K}$, where $\pi_\eta \colon \eta \to \operatorname{Spec} K$ is the natural morphism. Similarly, for $\sigma \in X(k)$, the action of $G_\sigma$ over $\mathbb{Z}$ is trivial and $\pi_{\sigma *}\mathbb{Z} = \mathbb{Z}$, where $\pi_\sigma \colon \sigma \to \operatorname{Spec} k$.

## 4. The Weil-divisor exact sequence

Throughout this section we will use the notations of Section 3.

4.1. **Proposition.** *There exists an exact sequence of sheaves on $X$ (with the étale topology)*

$$0 \to \mathbb{G}_{m,X} \xrightarrow{\alpha} g_*\mathbb{G}_{m,\eta} \xrightarrow{\beta} \bigoplus_{\sigma \in X(k)} \sigma_*\mathbb{Z}_\sigma \to 0; \tag{2}$$

*$\alpha$ and $\beta$ are defined as follows: if $W \xrightarrow{f} X$ is an étale morphism from a connected scheme and $R(W)$ is the ring of rational functions on $W$, then $\Gamma(W, \mathbb{G}_m) \xrightarrow{\alpha(W,f)} R(W)^*$ is induced by the morphism $W \times_X \eta \to W$ and, for all $\lambda \in R(W)^*$,*

$$\beta(W, f)(\lambda) = \sum_{w \in W(k)} \mathrm{v}_w(\lambda),$$

*where $\mathrm{v}_w$ is the discrete valuation on $R(W)$ induced by $\mathscr{O}_{W,w}$.*

*Proof.* If $W$ is a scheme as in the statement, then it is normal and regular. Moreover

$$\begin{aligned} \mathbb{G}_{m,X}(W, f) &= \Gamma(W, \mathbb{G}_m) \\ g_*\mathbb{G}_{m,\eta}(W, f) &= \Gamma(W \times_X \eta, \mathbb{G}_m) = R(W)^* \\ \bigoplus_{\sigma \in X(k)} \sigma_*\mathbb{Z}_\sigma(W, f) &= \bigoplus_{\sigma \in X(k)} \Gamma(W \times_X \sigma, \mathbb{Z}) = \bigoplus_{w \in W(k)} \mathbb{Z}. \end{aligned}$$

Therefore $\alpha(W, f)$ and $\beta(W, f)$ are well-defined. If $Y \xrightarrow{g} X$ is an other étale morphism from a scheme, we form the fiber product $h\colon Y \times_X W \to X$. Then the maps induced by restrictions on $Y \times_X W$ by $\alpha(W, f)$ and $\alpha(Y, g)$ coincide with $\alpha(Y \times_X W, h)$. A similar argument holds for $\beta$. It follows that $\alpha$ and $\beta$ are well-defined.

Recall that the exactness of a sequence of sheaves on a stack can be checked on stalks at geometric points. Let $\overline{x} \xrightarrow{x} X$ be a geometric point and let $f\colon U \to X$ be an étale morphism from a smooth connected curve, such that $x$ farctors through $f$. Then

$$\begin{aligned} (\mathbb{G}_{m,X})_{\overline{x}} &= (\mathbb{G}_{m,U})_{\overline{x}} = A^*, \\ (g_*\mathbb{G}_{m,\eta})_{\overline{x}} &= (g_{U*}\mathbb{G}_{m,\eta_U})_{\overline{x}} = Q(A)^*, \end{aligned}$$

where $A = \mathscr{O}^{\mathrm{sh}}_{U,\overline{x}}$ (note that $A$ is a discrete valuation ring), $Q(A)$ is its field of fractions, the morphism $g_U\colon \eta_U \to U$ is the inclusion of the generic point of $U$ (since $g$ is an open immersion and $U$ is irreducible, we obtain $U \times_X \eta = \eta_U$). Moreover, for every $\sigma \in X(k)$, $\sigma \times_X U$ is a set of closed points of $U$ whose image in $X$ is $\sigma$. Hence $(\sigma_*\mathbb{Z}_\sigma) = \bigoplus_{u \in \sigma \times_X U} u_*\mathbb{Z}$ and $(u_*\mathbb{Z})_{\overline{x}}$ is non zero if and only if $u = x$, in which case $(u_*\mathbb{Z})_{\overline{u}} = \mathbb{Z}$. Therefore the sequence of stalks is

$$0 \to A^* \to Q(A)^* \xrightarrow{\mathrm{v}_A} \mathbb{Z} \to 0,$$

where $\mathrm{v}_A$ is the discrete valuation on $Q(A)$ defined by $A$; this sequence is exact by [8], II.3.9. $\square$

4.2. The short exact sequence (2) induces a long exact sequence of étale cohomology groups

$$(3) \quad \cdots \to H^r_{\text{ét}}(X, \mathbb{G}_{m,X}) \to H^r_{\text{ét}}(X, g_*\mathbb{G}_{m,\eta}) \to \bigoplus_{\sigma \in X(k)} H^r_{\text{ét}}(X, \sigma_*\mathbb{Z}_\sigma) \to H^{r+1}_{\text{ét}}(X, \mathbb{G}_{m,X}) \to \cdots$$

where we used the fact that étale cohomology commutes with arbitrary direct sums of sheaves on a quasi-compact Deligne-Mumford stack (see [8], chapter III, Remark 3.6(d)).

4.3. Remark. By [8], III.2.22, there is a short exact sequence in cohomology

$$(4) \qquad 0 \to \Gamma(C, \mathbb{G}_m) \to K^* \xrightarrow{\beta_0} \bigoplus_{x \in C(k)} \mathbb{Z} \to \mathrm{Pic}(C) \to 0,$$

obtained from the Weil-divisor exact sequence for $C$ ([8], II.3.9).

4.4. **Proposition.** *Let $L$ be a quasi-algebraically closed field and let $G$ be a finite group of order not divided by the characteristic of $L$, acting trivially on $L$. Let $F$ be a sheaf over the quotient stack $[{}^{\operatorname{Spec} L}/_{G}]$. If either $L$ is algebraically closed or $F = \mathbb{G}_m$ then, for $r \geq 0$,*

$$H^r_{\acute{e}t}([{}^{\operatorname{Spec} L}/_{G}], F) \cong H^r(G, \Gamma(\operatorname{Spec} L, F)).$$

*Proof.* The natural map $\operatorname{Spec} L \to [{}^{\operatorname{Spec} L}/_{G}]$ is a Galois covering with group $G$, then we can consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q_{\text{ét}}(\operatorname{Spec} L, F)) \Rightarrow H^{p+q}_{\text{ét}}([{}^{\operatorname{Spec} L}/_{G}], F).$$

We can view the groups $H^r_{\text{ét}}(\operatorname{Spec} L, F)$ as Galois cohomology groups of the absolute Galois group of $L$ ([8], III.1.7). If $L$ is algebraically closed then $H^r_{\text{ét}}(\operatorname{Spec} L, F) = 0$ for $r \geq 1$. If $F = \mathbb{G}_m$ then, by Corollary 2.3, $H^r_{\text{ét}}(\operatorname{Spec} L, \mathbb{G}_m) = 0$ for $r \geq 1$. In both cases the sequence degenerates and we get the statement. □

4.5. **Lemma.** *Let $\sigma \in X(k)$ with stabilizer $G_\sigma$. Then $H^r_{\acute{e}t}(X, \sigma_* \mathbb{Z}_\sigma) = H^r(G_\sigma, \mathbb{Z})$, for $r \geq 0$.*

*Proof.* Consider the Leray spectral sequence for the inclusion $\sigma \xrightarrow{\sigma} X$,

$$E_2^{p,q} = H^p_{\text{ét}}(X, R^q \sigma_* \mathbb{Z}_\sigma) \Rightarrow H^{p+q}_{\text{ét}}(\sigma, \mathbb{Z}).$$

Since $\sigma$ is a closed embedding, the functor $\sigma_*$ is exact ([8], II.3.6), hence $R^q \sigma_* \mathbb{Z}_\sigma = 0$ for $q \geq 1$. Therefore the spectral sequence degenerates and $H^r_{\text{ét}}(X, \sigma_* \mathbb{Z}_\sigma) \cong H^r_{\text{ét}}(\sigma, \mathbb{Z})$ for $r \geq 0$. By Theorem 3.3, $\sigma$ is a gerbe over $\operatorname{Spec} k$ banded by $\mathscr{G}_\sigma$, and recall that such gerbes are classified by $H^2_{\text{ét}}(\operatorname{Spec} k, \mathscr{G}_\sigma)$, which vanishes since $k$ is algebraically closed. It follows that $\sigma$ is the trivial $G_\sigma$-gerbe over $\operatorname{Spec} k$ and, by Proposition 4.4, $H^r_{\text{ét}}(\sigma, \mathbb{Z}) \cong H^r(G_\sigma, \mathbb{Z})$ for $r \geq 0$. □

4.6. **Lemma.** *Let $G_0$ denote the stabilizer of $\eta$. Then $H^r_{\acute{e}t}(X, g_* \mathbb{G}_{m,\eta}) = H^r(G_0, K^*)$, for $r \geq 0$.*

*Proof.* Consider the Leray spectral sequence for $\eta \xrightarrow{g} X$,

$$E_2^{p,q} = H^p_{\text{ét}}(X, R^q g_* \mathbb{G}_{m,\eta}) \Rightarrow H^{p+q}_{\text{ét}}(\eta, \mathbb{G}_m).$$

Let $\overline{x} \xrightarrow{x} X$ be a geometric point and let $f \colon U \to X$ be an étale morphism from a connected curve, such that $\overline{x}$ farctors through $f$. Then $\eta \times_X U$ is the generic point $\eta_U$ of $U$. Therefore

$$(R^q g_* \mathbb{G}_{m,\eta})_{\overline{x}} = (R^q g_{U*} \mathbb{G}_{m,\eta_U})_{\overline{x}} = H^q_{\text{ét}}(\operatorname{Spec} \mathscr{O}^{\text{sh}}_{U,\overline{x}} \times_U \eta_U, \mathbb{G}_m) = H^q_{\text{ét}}(\operatorname{Spec} Q(\mathscr{O}^{\text{sh}}_{U,\overline{x}}), \mathbb{G}_m) = 0$$

for $q \geq 1$, where $Q(\mathscr{O}^{\text{sh}}_{U,\overline{x}})$ is the field of fractions of $\mathscr{O}^{\text{sh}}_{U,\overline{x}}$ and the last equality follows from the fact that $Q(\mathscr{O}^{\text{sh}}_{U,\overline{x}})$ is quasi-algebraically closed (Theorem 2.5). Hence $R^q g_* \mathbb{G}_{m,\eta} = 0$ for $q \geq 1$ and $H^r_{\text{ét}}(X, g_* \mathbb{G}_{m,\eta}) \cong H^r_{\text{ét}}(\eta, \mathbb{G}_m)$ for $r \geq 0$. Since $\eta = [{}^{\operatorname{Spec} K}/_{G_0}]$ is the trivial gerbe over $\operatorname{Spec} K$ (Proposition 3.5), the statement follows by Proposition 4.4. □

4.7. By Lemma 4.5 and Lemma 4.6, the sequence (3) becomes

$$(5) \quad \cdots \to H^r_{\text{ét}}(X, \mathbb{G}_m) \to H^r(G_0, K^*) \xrightarrow{\beta^{(r)}} \bigoplus_{\sigma \in X(k)} H^r(G_\sigma, \mathbb{Z}) \to H^{r+1}_{\text{ét}}(X, \mathbb{G}_m) \to \cdots.$$

4.8. **Lemma.** *For every $r \geq 0$, the map $\beta^{(r)}$ in (5) factors as*

$$H^r(G_0, K^*) \xrightarrow{\beta_0^{(r)}} \bigoplus_{\sigma \in X(k)} H^r(G_0, \mathbb{Z}) \xrightarrow{\tau^{(r)}} \bigoplus_{\sigma \in X(k)} H^r(G_\sigma, \mathbb{Z}),$$

*where $\beta_0^{(r)}$ is induced by $\beta_0$ in (4) and $\tau^{(r)}$ is the transfer map ([11], 6.7.16) on each component.*

*Proof.* Let us notice that $H^1(G_\sigma,\mathbb{Z})=\operatorname{Hom}(G_\sigma,\mathbb{Z})=0$, since $G_\sigma$ acts trivially on $\mathbb{Z}$ and $\mathbb{Z}$ does not contain non-trivial finite subgroups. From (4) and (5), we get the following commutative diagram with exact rows (by Remark 3.6, $\pi^*\mathbb{G}_{m,C}=\mathbb{G}_{m,X}$ and $\pi_\sigma^*\mathbb{Z}=\mathbb{Z}$)

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(C,\mathbb{G}_m) \arrow[r] \arrow[d, "\pi^*"] & K^* \arrow[r, "\beta_0"] \arrow[d, "\pi_\eta^*"] & \bigoplus_{\sigma\in X(k)} \mathbb{Z} \arrow[r] \arrow[d, "(\pi_\sigma^*)_\sigma"] & \operatorname{Pic}(C) \arrow[r] \arrow[d, "\pi^*"] & 0 \arrow[d] \\
0 \arrow[r] & H^0_{\text{ét}}(X,\mathbb{G}_m) \arrow[r] & K^* \arrow[r, "\beta"] & \bigoplus_{\sigma\in X(k)} \mathbb{Z} \arrow[r] & H^1_{\text{ét}}(X,\mathbb{G}_m) \arrow[r] & H^1(G_0,K^*) \arrow[r] & 0
\end{tikzcd}$$

Elements in $\Gamma(\operatorname{Spec}K,\mathbb{G}_m)$ corresponds to rational maps $C\to\mathbb{P}^1$. Since $C$ is smooth, we obtain $\Gamma(\operatorname{Spec}K,\mathbb{G}_m)\subset\operatorname{Mor}(C,\mathbb{P}^1)$. Similarly, $H^0_{\text{ét}}(\eta,\mathbb{G}_m)\subset\operatorname{Mor}(X,\mathbb{P}^1)$. Because $C$ is the coarse moduli space, $\operatorname{Mor}(C,\mathbb{P}^1)\cong\operatorname{Mor}(X,\mathbb{P}^1)$ and under this identification $\pi_\eta^*=\mathrm{id}$.

Recall that $\pi$ factors through an étale gerbe $X\to Y$ over an orbicurve $Y$. Let $\beta_Y$ be the analogous of $\beta$ for $Y$, then $\beta=\beta_Y$, since the valuation at a point doesn't change for étale morphisms. Hence we can assume $X=Y$. As in the proof of Proposition 3.1, étale locally $Y=[{}^{\operatorname{Spec}k[t]}/_{\mu_r}]$ and $C=\operatorname{Spec}k[t^r]$. Considering the étale cover $\operatorname{Spec}k[t]\to Y$, we see that $\pi_\sigma^*$ is the multiplication by the order of the stabilizer of $\sigma\in Y(k)$ in $Y$. Let $d_\sigma$ be the order of $G_\sigma$ and $d$ the order of $G_0$, then $\beta=({}^{d_\sigma}/_d)_\sigma\circ\beta_0$ and $\beta^{(r)}$ is induced by $\beta_0$ and the inclusion $G_0\hookrightarrow G_\sigma$. □

4.9. Remark. By Lemma 4.8 and snake lemma, we have an exact sequence

$$0\to\ker\beta_0^{(r)}\to\ker\beta^{(r)}\to\ker\tau^{(r)}\to\operatorname{coker}\beta_0^{(r)}\to\operatorname{coker}\beta^{(r)}\to\operatorname{coker}\tau^{(r)}\to 0.$$

Moreover, the sequence (4) induces exact sequences in cohomology (notice that $\operatorname{im}\beta_0$ is a free $\mathbb{Z}$-module)

$$\begin{cases} 0\to H^r(G_0,\Gamma(C,\mathbb{G}_m))\to H^r(G_0,K^*)\xrightarrow{\widetilde{\beta}_0^{(r)}} H^r(G_0,\operatorname{im}\beta_0)\to 0,\\ \cdots\to H^r(G_0,\operatorname{im}\beta_0)\to\bigoplus_{x\in C(k)}H^r(G_0,\mathbb{Z})\to H^r(G_0,\operatorname{Pic}(C))\to\cdots. \end{cases}$$

By snake lemma, we get $\operatorname{coker}\beta_0^{(r)}\subset H^r(G_0,\operatorname{Pic}(C))$ and $H^r(G_0,\Gamma(C,\mathbb{G}_m))\subset\ker\beta_0^{(r)}$.

4.10. **Theorem.** *With the notations of Section 3, we have $H^0_{\acute{e}t}(X,\mathbb{G}_m)=\Gamma(C,\mathbb{G}_m)$ and, for $r\geq 1$, there exists a short exact sequence*

$$0\to\operatorname{coker}\gamma^{(r-1)}\to H^r_{\acute{e}t}(X,\mathbb{G}_m)\to\ker\beta^{(r)}\to 0,$$

*where $\ker\beta^{(r)}$ fits in the following exact sequence*

$$0\to H^r(G_0,\Gamma(C,\mathbb{G}_m))\to\ker\beta^{(r)}\to\ker\gamma^{(r)}\to 0$$

*and $\gamma^{(r)}\colon H^r(G_0,\operatorname{im}\beta_0)\to\bigoplus_{\sigma\in X(k)}H^r(G_\sigma,\mathbb{Z})$ is induced by $G_0\hookrightarrow G_\sigma$ and the natural inclusion $\operatorname{im}\beta_0\subset\bigoplus_{\sigma\in X(k)}\mathbb{Z}$. In particular there is a short exact sequence*

$$0\to\operatorname{Pic}(Y)\to H^1_{\acute{e}t}(X,\mathbb{G}_m)\to \mathit{Hom}(G_0,K^*)\to 0.$$

*Proof.* By sequence (5), we have short exact sequences

$$0\to\operatorname{coker}\beta^{(r-1)}\to H^r_{\text{ét}}(X,\mathbb{G}_m)\to\ker\beta^{(r)}\to 0$$

for $r\geq 1$. By Remark 4.9, $H^0_{\text{ét}}(X,\mathbb{G}_m)=H^0_{\text{ét}}(C,\mathbb{G}_m)$ and $\beta^{(r)}=\gamma^{(r)}\circ\widetilde{\beta}_0^{(r)}$, with $\gamma^{(r)}$ as described in the statement. Applying snake lemma we get $\operatorname{coker}\beta^{(r-1)}=\operatorname{coker}\gamma^{(r-1)}$ and

$$0\to H^r(G_0,\Gamma(C,\mathbb{G}_m))\to\ker\beta^{(r)}\to\ker\gamma^r\to 0.$$

As noticed in the proof of Lemma 4.8, $\operatorname{coker}\beta = \operatorname{Pic}(Y)$, hence we obtain

$$0 \to \operatorname{Pic}(Y) \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to \operatorname{Hom}(G_0, K^*) \to 0,$$

where we used that $H^1(G_\sigma, \mathbb{Z}) = \operatorname{Hom}(G_\sigma, \mathbb{Z}) = 0$, since $G_\sigma$ acts trivially on $\mathbb{Z}$. $\square$

4.11. Remark. In the case $G_0$ is abelian and $X$ is a $G_0$-gerbe banded over $Y$, we recover the description of $H^1_{\text{ét}}(X, \mathbb{G}_m)$ given in [4], Corollary 3.2.1 (see also Section 5.1).

4.12. **Proposition.** *If $G_\sigma$ is abelian for every $\sigma \in X(k)$ then $\ker\beta^{(r)}$ depends only on $Y$ and $G_0$, not on the structure of the gerbe $X \to Y$. In particular $H^2_{\text{ét}}(X, \mathbb{G}_m) = H^2_{\text{ét}}(Y \times \mathrm{B}G_0, \mathbb{G}_m)$.*

*Proof.* By Theorem 4.10, $H^2_{\text{ét}}(X, \mathbb{G}_m) = \ker\beta^{(2)}$. Consider the restriction map

$$\operatorname{res}^{(r)}\colon H^r(G_{\sigma_l}, \mathbb{Z}) \to H^r(G_0, \mathbb{Z}),$$

where $G_0$ and $G_{\sigma_l}$ act trivially on $\mathbb{Z}$. By Hochschild-Serre spectral sequence ([11], 6.8.2)

$$H^p(\mathbb{Z}/d_l\mathbb{Z}, H^q(G_0, \mathbb{Z})) \Rightarrow H^{p+q}(G_{\sigma_l}, \mathbb{Z})$$

and Lemma 2.9, we see that $\operatorname{res}^{(r)}$ is surjective. Recall that $\tau^{(r)} \circ \operatorname{res}^{(r)} = d_l$ ([11], Lemma 6.7.17), where $H^r(G_{\sigma_l}, \mathbb{Z}) \xrightarrow{d_l} H^r(G_{\sigma_l}, \mathbb{Z})$ is the multiplication by $d_l$ which is induced by $\mathbb{Z} \xrightarrow{d_l} \mathbb{Z}$. It follows that $\ker\tau^{(r)} = \operatorname{im}\operatorname{res}^{(r)}|_{\ker d_l}$. The same argument applies to $Y \times \mathrm{B}G_0$. Notice that

$$\tau_Y^{(r)}\colon H^r(G_0, \mathbb{Z}) \to H^r(G_0 \times \mathbb{Z}/d_l\mathbb{Z}, \mathbb{Z})$$

is induced by composition with the projection $G_0 \times \mathbb{Z}/d_l\mathbb{Z} \to G_0$ and the multiplication by $d_l$, therefore $\ker d_l \subset \ker\tau_Y^{(r)}$. As a consequence $\ker\tau^{(r)} \subset \ker\tau_Y^{(r)}$ and hence $\ker\beta^{(r)} \subset \ker\beta_Y^{(r)}$, where $\beta_Y^{(r)} = \tau_Y^{(r)} \circ \beta_0^{(r)}$. By Remark 4.9, we get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker\beta_0^{(r)} \arrow[r] \arrow[d, equal] & \ker\beta^{(r)} \arrow[r] \arrow[d, hook] & \ker\tau^{(r)} \arrow[r] \arrow[d, hook] & \operatorname{coker}\beta_0^{(r)} \arrow[d, equal] \\
0 \arrow[r] & \ker\beta_0^{(r)} \arrow[r] & \ker\beta_Y^{(r)} \arrow[r] & \ker\tau_Y^{(r)} \arrow[r] \arrow[d, hook] & \operatorname{coker}\beta_0^{(r)} \arrow[d, hook] \\
 & & & \bigoplus_{x \in C(k)} H^r(G_0, \mathbb{Z}) \arrow[r] & H^r(G_0, \operatorname{Pic}(C))
\end{tikzcd}$$

which implies $\ker\beta^{(r)} = \ker\beta_Y^{(r)}$. $\square$

4.13. Remark. In general, the groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ depend on the gerbe $X \to Y$; if the stabilizers are not abelian, also $H^2_{\text{ét}}(X, \mathbb{G}_m)$ may depend on $X \to Y$ (see Section 7).

4.14. **Corollary.** *If $G_\sigma$ is cyclic for every $\sigma \in X(k)$ then, for every $r \geq 1$,*

$$H^{2r}_{\text{ét}}(X, \mathbb{G}_m) = H^2_{\text{ét}}(Y \times \mathrm{B}G_0, \mathbb{G}_m)$$

*and there is a short exact sequence*

$$0 \to H^2(G_0, \operatorname{Pic}(Y)) \to H^{2r+1}_{\text{ét}}(X, \mathbb{G}_m) \to Q \to 0,$$

*where $Q$ fits in the following exact sequence*

$$0 \to \bigoplus_{l=1}^{N} \mathbb{Z}/d_l\mathbb{Z} \to Q \to \mathit{Hom}(G_0, K^*) = G_0 \to 0.$$

*Proof.* By Corollary 2.8 and Theorem 2.7, the sequence (5) induces exact sequences for $r \geq 1$,

$$0 \to H^{2r}_{\text{ét}}(X, \mathbb{G}_m) \to {}^{K^*}\!/\!{}_{K^{*d}} \xrightarrow{\beta^{(2)}} \bigoplus_{\sigma \in X(k)} G_\sigma \to H^{2r+1}_{\text{ét}}(X, \mathbb{G}_m) \to G_0 \to 0.$$

In particular $H^{2r}_{\text{ét}}(X, \mathbb{G}_m) = H^2_{\text{ét}}(X, \mathbb{G}_m)$ and $H^2_{\text{ét}}(X, \mathbb{G}_m) = H^2_{\text{ét}}(Y \times \mathrm{B}G_0, \mathbb{G}_m)$, by Proposition 4.12. Moreover, by Lemma 4.8, the map $\beta^{(2)}$ factors as

$$H^2(G_0, K^*) \xrightarrow{\overline{\beta}} \bigoplus_{\sigma \in X(k)} H^2(G_0, \mathbb{Z}) = G_0 \hookrightarrow \bigoplus_{\sigma \in X(k)} G_\sigma = H^2(G_\sigma, \mathbb{Z}),$$

where $\overline{\beta} = ({}^{d_\sigma}\!/\!{}_{d}) \circ \beta_0^{(2)}$ is the map induced by $\beta$. By Remark 4.9, there is an exact sequence

$$H^2(G_0, \operatorname{im}\beta) \to \bigoplus_{\sigma \in X(k)} H^2(G_0, \mathbb{Z}) \to H^2(G_0, \operatorname{Pic}(Y)) \to H^3(G_0, \operatorname{im}\beta) = 0,$$

which implies $\operatorname{coker}\overline{\beta} = H^2(G_0, \operatorname{Pic}(Y))$. By snake lemma, we get the following exact sequence

$$0 \to H^2(G_0, \operatorname{Pic}(Y)) \to \operatorname{coker}\beta^{(2)} \to \bigoplus_{l=1}^{N} {}^{\mathbb{Z}}\!/\!{}_{d_l\mathbb{Z}} \to 0.$$

□

4.15. **Corollary.** *Let $Y$ be a smooth tame orbicurve over an algebraically closed field, then*

$$H^r_{\acute{e}t}(Y, \mathbb{G}_m) = \begin{cases} \Gamma(C, \mathbb{G}_m) & r = 0 \\ \operatorname{Pic}(Y) & r = 1 \\ 0 & r \equiv 0 \quad (2),\ r \geq 2 \\ \bigoplus_{l=1}^{N} {}^{\mathbb{Z}}\!/\!{}_{d_l\mathbb{Z}} & r \equiv 1 \quad (2),\ r \geq 3. \end{cases}$$

*Moreover, there is a short exact sequence*

$$0 \to \operatorname{Pic}(C) \to \operatorname{Pic}(Y) \to \bigoplus_{l=1}^{N} {}^{\mathbb{Z}}\!/\!{}_{d_l\mathbb{Z}} \to 0.$$

*Proof.* By the proof of Lemma 4.8, we get the description of $H^1_{\text{ét}}(Y, \mathbb{G}_m)$. Moreover, applying Theorem 4.10 with $G_0 = 0$, we have $H^r_{\text{ét}}(Y, \mathbb{G}_m) = \bigoplus_{l=1}^{N} H^{r-1}({}^{\mathbb{Z}}\!/\!{}_{d_l\mathbb{Z}}, \mathbb{Z})$, for $r \geq 2$, and the statement follows from Corollary 2.8. □

## 5. Twisted nodal curves

Let $Y$ be a twisted nodal curve over an algebraically closed field $k$ ([2], Definition 4.1.2). In particular $Y$ is a connected tame Deligne-Mumford stack over $k$ with trivial generic stabilizer. If $C$ is the coarse moduli space of $Y$, then $C$ is a connected nodal curve over $k$. Let $\pi\colon Y \to C$ be the natural morphism; we denote by $S$ the set of singular points of $C$ and we write $S_Y = S \times_C Y$.

5.1. **Proposition.** *We have $H^r_{\acute{e}t}(C, \mathbb{G}_m) = 0$ for $r \geq 2$.*

*Proof.* Let $\nu\colon \hat{C} \to C$ be the normalization of $C$, then, by Theorem 3.2, $H^r_{\text{ét}}(\hat{C}, \mathbb{G}_m) = 0$ for $r \geq 2$. Moreover $\nu$ is a finite morphism, therefore $\nu_*$ is an exact functor ([8], II.3.6), and by Leray spectral sequence for $\nu$, we get $H^r_{\text{ét}}(C, \nu_*\mathbb{G}_{m,\hat{C}}) \cong H^r_{\text{ét}}(\hat{C}, \mathbb{G}_m)$ for $r \geq 0$. There is a natural injective morphism of sheaves $\mathbb{G}_{m,C} \to \nu_*\mathbb{G}_{m,\hat{C}}$, whose cokernel is concentrated in the singular locus of $C$. Equivalently we have the following exact sequence

$$0 \to \mathbb{G}_{m,C} \to \nu_*\mathbb{G}_{m,\hat{C}} \to \bigoplus_{x \in S} x_* Q_x \to 0,$$

where $Q_x$ is a sheaf over $x = \operatorname{Spec} k$. Consider the long exact sequence in cohomology

$$\cdots \to H^r_{\text{ét}}(C, \mathbb{G}_m) \to H^r_{\text{ét}}(C, \nu_* \mathbb{G}_{m,\hat{C}}) \to \bigoplus_{x \in S} H^r_{\text{ét}}(C, x_* Q_x) \to \cdots.$$

Since $x\colon \operatorname{Spec} k \to C$ is a closed embedding, the functor $x_*$ is exact ([8], II.3.6) and by Leray spectral sequence for $x$, we get $H^r_{\text{ét}}(C, x_* Q_x) \cong H^r_{\text{ét}}(\operatorname{Spec} k, Q_x)$ for $r \geq 0$. Finally, the groups $H^r_{\text{ét}}(\operatorname{Spec} k, Q_x)$ vanish for $r \geq 1$, since $k$ is algebraically closed. □

5.2. **Proposition.** *Let $\Sigma$ be the set of closed points of $Y$ with non trivial stabilizer. Then*

$$H^r_{\acute{e}t}(Y, \mathbb{G}_m) = \begin{cases} \Gamma(C, \mathbb{G}_m) & r = 0 \\ \operatorname{Pic}(Y) & r = 1 \\ 0 & r \equiv 0 \quad (2),\ r \geq 2 \\ \bigoplus_{\sigma \in \Sigma} \mathbb{Z}/d_\sigma\mathbb{Z} & r \equiv 1 \quad (2),\ r \geq 3. \end{cases}$$

*Moreover, there is a short exact sequence*

$$0 \to \operatorname{Pic}(C) \to \operatorname{Pic}(Y) \to \bigoplus_{\sigma \in \Sigma} \mathbb{Z}/d_\sigma\mathbb{Z} \to 0.$$

*Proof.* With notations as in the proof of Proposition 5.1, let $\hat{Y} = Y \times_C \hat{C}$. Notice that the induced morphism $\hat{\nu}\colon \hat{Y} \to Y$ is finite because $\nu$ is finite, hence, by Leray spectral sequence for $\hat{\nu}$, we get $H^r_{\text{ét}}(Y, \hat{\nu}_* \mathbb{G}_{m,\hat{Y}}) = H^r_{\text{ét}}(\hat{Y}, \mathbb{G}_m)$ for $r \geq 0$. Moreover $\hat{Y}$ is smooth, hence Corollary 4.15 gives a description of the groups $H^r_{\text{ét}}(\hat{Y}, \mathbb{G}_m)$. We have the following exact sequence of sheaves

$$0 \to \mathbb{G}_{m,Y} \to \hat{\nu}_* \mathbb{G}_{m,\hat{Y}} \to Q \to 0,$$

where $Q = \bigoplus_{\sigma \in S_Y} \sigma_* Q_\sigma$ is concentrated in $S_Y$. Consider the long exact sequence in cohomology

$$\cdots \to H^r_{\text{ét}}(Y, \mathbb{G}_m) \to H^r_{\text{ét}}(Y, \hat{\nu}_* \mathbb{G}_{m,\hat{Y}}) \to \bigoplus_{\sigma \in S_Y} H^r_{\text{ét}}(Y, \sigma_* Q_\sigma) \to \cdots. \tag{6}$$

Since $\sigma$ is a closed embedding, by Leray spectral sequence for $\sigma$ and Proposition 4.4, we obtain $H^r_{\text{ét}}(Y, \sigma_* Q_\sigma) \cong H^r_{\text{ét}}(\sigma, Q_\sigma) = H^r(G_\sigma, Q^{\text{sh}}_\sigma)$ for $r \geq 0$, where $G_\sigma = \mathbb{Z}/d_\sigma\mathbb{Z}$ is the stabilizer of $\sigma$ and $Q^{\text{sh}}_\sigma$ is the strictly henselian local ring of $Q$ at $\sigma$. Notice that $Q^{sh}_\sigma = \hat{A}^*/A^*$, where $A = \mathscr{O}^{\text{sh}}_{Y,\sigma}$ and $\operatorname{Spec} \hat{A}$ is the normalization of $\operatorname{Spec} A$. Since $Y$ is étale locally a nodal curve over $k$, we have that $\hat{A}^* = k^* \times k^*$ and $A^* = k^*$, therefore $Q^{\text{sh}}_\sigma = k^*$. By Theorem 2.7, we have

$$H^r(G_\sigma, Q^{\text{sh}}_\sigma) = \begin{cases} k^* & r = 0 \\ \mathbb{Z}/d_\sigma\mathbb{Z} & r \equiv 1 \quad (2) \\ 0 & r \equiv 0 \quad (2),\ r \geq 2. \end{cases}$$

We substitute these results in (6) and obtain, for $r \geq 1$,

$$0 \to H^{2r+1}_{\text{ét}}(Y, \mathbb{G}_m) \to \bigoplus_{\sigma \in \hat{\Sigma}} \mathbb{Z}/d_\sigma\mathbb{Z} \xrightarrow{\rho} \bigoplus_{\sigma \in S_Y} \mathbb{Z}/d_\sigma\mathbb{Z} \to H^{2r+2}_{\text{ét}}(Y, \mathbb{G}_m) \to 0,$$

where $\hat{\Sigma}$ is the set of closed points of $\hat{Y}$ with non trivial stabilizer and the map $\rho$ is described as follows. If $\sigma \in S_Y$, with $d_\sigma > 0$, then $\sigma \times_Y \hat{Y}$ consists of two points $\sigma_1, \sigma_2 \in \hat{\Sigma}$; let $\rho_\sigma$ be the restriction of $\rho$ to $\mathbb{Z}/d_{\sigma_1}\mathbb{Z} \oplus \mathbb{Z}/d_{\sigma_2}\mathbb{Z}$, then $\rho_\sigma(a, b) = a - b \in \mathbb{Z}/\sigma\mathbb{Z}$. Otherwise, the restriction of $\rho$ to $\mathbb{Z}/d_\sigma\mathbb{Z}$ is zero. It follows that, for every $r \geq 3$,

$$H^r_{\text{ét}}(Y, \mathbb{G}_m) = \begin{cases} \bigoplus_{\sigma \in \Sigma} \mathbb{Z}/d_\sigma\mathbb{Z} & r \equiv 1 \quad (2) \\ 0 & r \equiv 0 \quad (2). \end{cases}$$

Moreover we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(C,\mathbb{G}_m) \arrow[r] \arrow[d] & \Gamma(\hat{C},\mathbb{G}_m) \arrow[r, "\rho_0"] \arrow[d, equal] & \bigoplus_{\sigma\in S_Y} k^* \arrow[r] \arrow[d, equal] & \mathrm{Pic}(C) \arrow[r] \arrow[d] & \mathrm{Pic}(\hat{C}) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^0_{\text{ét}}(Y,\mathbb{G}_m) \arrow[r] & \Gamma(\hat{C},\mathbb{G}_m) \arrow[r, "\rho_0"] & \bigoplus_{\sigma\in S_Y} k^* \arrow[r] & \mathrm{Pic}(Y) \arrow[r] & \mathrm{Pic}(\hat{Y}) \arrow[r] & \bigoplus_{\sigma\in S_Y} \mathbb{Z}/d_\sigma\mathbb{Z} \arrow[r] & H^2_{\text{ét}}(Y,\mathbb{G}_m) \arrow[r] & 0
\end{tikzcd}$$

which implies $H^0_{\text{ét}}(Y,\mathbb{G}_m)=\Gamma(C,\mathbb{G}_m)$ and $H^2_{\text{ét}}(Y,\mathbb{G}_m)=0$. Finally, by snake lemma, the following sequence is exact

$$0\to \mathrm{Pic}(C)\to H^1_{\text{ét}}(Y,\mathbb{G}_m)\to \bigoplus_{\sigma\in\Sigma}\mathbb{Z}/d_\sigma\mathbb{Z}\to 0. \qquad \square$$

5.1. **Banded gerbes over orbicurves.** Recall that, if $\mathscr{G}$ is a sheaf of abelian groups on a twisted nodal curve $Y$, then $H^2_{\text{ét}}(Y,\mathscr{G})$ classifies the $\mathscr{G}$-gerbes over $Y$ banded by $\mathscr{G}$ ([7], IV.3.5). Let $G_0$ be an abelian finite group of order not divided by $\operatorname{char} k$ and let $G_0=\bigoplus_{h=1}^M \mathbb{Z}/d_h\mathbb{Z}$ be a decomposition of $G_0$ as a direct sum of cyclic groups. We have a short exact sequence

$$0\to G_0\to \bigoplus_{h=1}^M \mathbb{G}_m\to \bigoplus_{h=1}^M \mathbb{G}_m\to 0,$$

deduced by Kummer sequence for each factor. From the induced long exact sequence in cohomology we get

$$\bigoplus_{h=1}^M \mathrm{Pic}(Y)\xrightarrow{\psi} H^2_{\text{ét}}(Y,G_0)=\bigoplus_{h=1}^M H^2_{\text{ét}}(Y,\mathbb{Z}/d_h\mathbb{Z})\to \bigoplus_{h=1}^M H^2_{\text{ét}}(Y,\mathbb{G}_m)=0,$$

where, according to [4] (Section 2.4), the map $\psi$ associates to $(L_1,\dots,L_M)\in\bigoplus_{l=1}^M \mathrm{Pic}(Y)$ the $G_0$-gerbe $\sqrt[d_1]{L_1/Y}\times_Y\cdots\times_Y \sqrt[d_M]{L_M/Y}$. Therefore every $G_0$-gerbe over $Y$ banded by $G_0$ is obtained as a finite number of root constructions.

## 6. Trivial gerbes

Throughout this section we will use the notations of Section 3.

6.1. **Proposition.** *We have $H^1_{\acute{e}t}(Y\times \mathrm{B}G_0,\mathbb{G}_m)=\mathrm{Pic}(Y)\oplus Hom(G_0,K^*)$ and, for $r\ge 2$, there are short exact sequences*

$$0\to Q^r\to H^r_{\acute{e}t}(Y\times \mathrm{B}G_0,\mathbb{G}_m)\to \bigoplus_{l=1}^N H^{r-1}(G_0\times \mathbb{Z}/d_l\mathbb{Z},\mathbb{Z})/H^{r-1}(G_0,\mathbb{Z})\to 0,$$

*where $Q^r$ fits in the following exact sequence*

$$0\to H^r(G_0,\Gamma(C,\mathbb{G}_m))\to Q^r\to H^{r-1}(G_0,\mathrm{Pic}(Y))\to 0.$$

*Proof.* The map $Y\xrightarrow{\rho} Y\times \mathrm{B}G_0$ obtained by $\operatorname{Spec} k\to \mathrm{B}G_0$ is a Galois cover with group $G_0$. Then we can consider the Hochschild-Serre spectral sequence

$$E_2^{p,q}=H^p(G_0,H^q_{\text{ét}}(Y,\mathbb{G}_m))\Rightarrow H^{p+q}_{\text{ét}}(Y\times \mathrm{B}G_0,\mathbb{G}_m). \tag{7}$$

By Corollary 4.15,

$$E_2^{p,q}=\begin{cases} H^p(G_0,\Gamma\, C,\mathbb{G}_m) & q=0\\ H^p(G_0,\mathrm{Pic}(Y)) & q=1\\ \bigoplus_{l=1}^N H^p(G_0,H^{q-1}(\mathbb{Z}/d_l\mathbb{Z},\mathbb{Z})) & q\ge 2.\end{cases}$$

In particular, the groups $E_2^{p,q}$ and the maps between them coincide, for $q \geq 2$, with the groups $F_2^{p,q-1}$ and relative maps for the Hochschild-Serre spectral sequence

$$F_2^{p,q} = \bigoplus_{l=1}^{N} H^p(G_0, H^q({}^{\mathbb{Z}}/_{d_l}\mathbb{Z}, \mathbb{Z})) \Rightarrow \bigoplus_{l=1}^{N} H^{p+q}(G_0 \times {}^{\mathbb{Z}}/_{d_l}\mathbb{Z}, \mathbb{Z}). \tag{8}$$

By Lemma 2.9, we have $E_\infty^{p,q} = E_2^{p,q}$. Comparing filtrations for (7) and (8), we get

$$0 \to H^r_{r-1} \to H^r_{\text{ét}}(Y \times \mathrm{B}G_0, \mathbb{G}_m) \to \bigoplus_{l=1}^{N} {}^{H^{r-1}({}^{\mathbb{Z}}/_{d_l}\mathbb{Z} \times G_0, \mathbb{Z})}\big/_{H^{r-1}(G_0,\mathbb{Z})} \to 0,$$

for $r \geq 1$, where $H^r_{r-1}$ fits in the following exact sequence

$$0 \to H^r(G_0, \Gamma(C, \mathbb{G}_m)) \to H^r_{r-1} \to H^{r-1}(G_0, \mathrm{Pic}(Y)) \to 0.$$

Consider the natural morphism $Y \times \mathrm{B}G_0 \xrightarrow{\pi} Y$. We have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{Pic}(Y) & \xleftarrow{\rho^*} & H^1_{\text{ét}}(Y \times \mathrm{B}G_0, \mathbb{G}_m) & \leftarrow & \mathrm{Hom}(G_0, K^*) & \longleftarrow & 0 \\
 & & {\scriptstyle (\pi\circ\rho)_*}\Big\| & & \Big\| & & \Big\| & & \\
0 & \longrightarrow & \mathrm{Pic}(Y) & \xrightarrow{\pi^*} & H^1_{\text{ét}}(Y \times \mathrm{B}G_0, \mathbb{G}_m) & \rightarrow & \mathrm{Hom}(G_0, K^*) & \longrightarrow & 0
\end{array}$$

where the second row is given by Theorem 4.10 and the first row is deduced from the argument above, after noticing that $\mathrm{Hom}(G_0, \Gamma(C, \mathbb{G}_m)) = \mathrm{Hom}(G_0, K^*)$. Hence we get

$$H^1_{\text{ét}}(Y \times \mathrm{B}G_0, \mathbb{G}_m) = \mathrm{Pic}(Y) \oplus \mathrm{Hom}(G_0, K^*).$$

□

6.2. Remark. If $G_0$ is cyclic then, by Theorem 2.7 and Proposition 6.1, we have, for $r \geq 1$,

$$H^{2r+1}_{\text{ét}}(C \times \mathrm{B}G_0, \mathbb{G}_m) = \mathrm{Hom}(G_0, \Gamma(C, \mathbb{G}_m)) \oplus H^2(G_0, \mathrm{Pic}(C)).$$

6.3. **Corollary.** *If $C$ is projective and $G_0$ is cyclic, then*

$$H^r_{\acute{e}t}(C \times \mathrm{B}G_0, \mathbb{G}_m) = \begin{cases} k^* & r = 0 \\ \mathrm{Pic}(C) \oplus G_0 & r = 1 \\ (G_0)^{2g} & r \equiv 0 \quad (2),\ r \geq 2 \\ G_0 \oplus G_0 & r \equiv 1 \quad (2),\ r \geq 3. \end{cases}$$

*Proof.* Let $d$ be the order of $G_0$. Recall that there is a short exact sequence

$$0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}(C) \to \mathbb{Z} \to 0.$$

Moreover, since $C$ is projective, $\Gamma(C, \mathbb{G}_m) = k^*$ and the sequence

$$0 \to ({}^{\mathbb{Z}}/_{d\mathbb{Z}})^{2g} \to \mathrm{Pic}^0(C) \xrightarrow{d} \mathrm{Pic}^0(C) \to 0$$

is exact, where the map $d$ is defined by $d(a) = a^d$, for all $a \in \mathrm{Pic}^0(C)$ (see [9], Section IV.21, Lang's Theorem). Therefore, applying Theorem 2.7, we obtain, for $r \geq 1$

$$\begin{cases} H^{2r-1}(G_0, k^*) = G_0, & H^{2r}(G_0, k^*) = 0, \\ H^{2r-1}(G_0, \mathrm{Pic}^0(C)) = (G_0)^{2g}, & H^{2r}(G_0, \mathrm{Pic}^0(C)) = 0, \\ H^{2r-1}(G_0, \mathrm{Pic}(C)) = (G_0)^{2g}, & H^{2r}(G_0, \mathrm{Pic}(C)) = G_0. \end{cases}$$

Then the statement follows from Proposition 6.1 and Remark 6.2. □

## 7. Examples

In this section we present two examples: the first one shows that, in general, the groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ may depend on the gerbe $X \to Y$, for $r \geq 3$; the second one shows that $H^2_{\text{ét}}(X, \mathbb{G}_m)$ may depend on the gerbe $X \to Y$ if the stabilizers are not abelian.

Let $p$ be a prime integer and consider $\mathbb{A}^1_{\mathbb{C}}$ with the action $\mathbb{C}[x] \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}[x]$ given by $(x, \lambda) \mapsto \lambda x$, for all $\lambda \in \mathbb{Z}/p\mathbb{Z}$. Then $Y = [\mathbb{A}^1_{\mathbb{C}}/\mu_p]$ is an orbicurve.

7.1. Let $X = [\mathbb{A}^1_{\mathbb{C}}/\mu_{p^2}]$, where the action is given by $(x, \lambda) \mapsto \lambda^p x$, for $\lambda \in \mathbb{Z}/p^2\mathbb{Z}$. By Kummer sequence, $H^2_{\text{ét}}(Y, \mu_p) = \mathbb{Z}/p\mathbb{Z}$ and, since $p$ is prime, there are only two non isomorphic banded $\mu_p$-gerbes over $Y$ ([7], IV.3.5); in particular, by Theorem 3.3, $X$ is the only non trivial $\mu_p$-gerbe banded over $Y$ (up to isomorphism). By Corollary 4.14, $H^{2r+1}_{\text{ét}}(X, \mathbb{G}_m)$ is finite of order $p^3$ whereas, by Proposition 6.1, $H^{2r+1}_{\text{ét}}(Y \times \mathrm{B}\mu_p, \mathbb{G}_m)$ has order $p^2$, for $r \geq 1$.

7.2. Set $p = 2$. Let $D_{2m}$ be the dihedral group with $2m$ elements, with $m$ odd, $m \geq 3$. We denote by $r \in \mathbb{Z}/m\mathbb{Z}$ and $s \in \mathbb{Z}/2\mathbb{Z}$ the generators of $D_{2m}$. Let $X = [\mathbb{A}^1_{\mathbb{C}}/D_{2m}]$, where the action is given by $(x, r^i s^\epsilon) \mapsto (-1)^\epsilon x$, for $1 \leq i \leq m$ and $\epsilon = 0, 1$. By [11], 6.7.10, we have $H^2(D_{2m}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, hence $\tau^{(2)} \colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the zero map. Therefore, by Remark 4.9 and Proposition 6.1, $H^2_{\text{ét}}(X, \mathbb{G}_m) = \ker \beta^{(2)} = \ker \tau^{(2)} = \mathbb{Z}/m\mathbb{Z}$ whereas $H^2_{\text{ét}}(Y \times \mathrm{B}\mu_m, \mathbb{G}_m) = 0$.

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